An Unconditionally Stable Finite Element Time-Domain Solution of the Vector Wave Equation

Stephen D. Gedney, *Member, IEEE,* and Umesh Navsariwala, *Student Member, IEEE*

Abstract—**This paper presents an implicit finite element time-domain (FETD) solution of the time-dependent vector wave equation. The time-dependent formulation employs a time-integration method based on the Newmark-Beta method. A stability analysis is presented demonstrating that this leads to an unconditionally stable solution of the time-dependent vector wave equation. The advantage of this formulation is that the time step is no longer governed by the spatial discretization of the mesh, but rather by the spectral content of the time-dependent signal. A numerical example of a three-dimensional cavity resonator is presented studying the effects of the Newmark-beta parameters on the solution error. Optimal choices of parameters are derived based on this example.**

I. Introduction

VOLUME DISCRETIZATION techniques have been used extensively for rigorous time-domain analyses of microwave circuits and resonators. The finite-difference time-domain (FDTD) method, an explicit method, has proven to be a highly efficient technique [1]. Finite element methods have also been implemented using both implicit [2]–[5], and explicit methods [6]. For an implicit method to be competitive with an explicit method, the number of time iterations required to converge to a final solution must be significantly less, since each time iteration requires a solution of a linear system of equations. Unfortunately, the implicit finite element time-domain (FETD) methods presented in [2]–[5] are conditionally stable, and in fact, stability requires time steps that can be smaller than those required by explicit FDTD methods [7]. In this letter, a FETD algorithm based on the Newmark Beta method [8], [9] is presented. It is shown that the algorithm is unconditionally stable for the three-dimensional vector FETD problem. Some numerical examples based on a cavity resonator problem are presented to demonstrate the efficiency and accuracy of the method and the optimal choice of parameters.

II. The Finite Element Formulation

Time-dependent electromagnetic fields radiated by an electric current density $\vec{J}$ are defined in a finite volume Ω bound by the surface $\partial\Omega$. The electric fields must satisfy the time-dependent inhomogeneous wave equation

$$\nabla \times \frac{1}{\mu_r}\nabla \times \vec{E} + \mu_0\sigma\frac{\partial \vec{E}}{\partial t} + \frac{\varepsilon_r}{c_0^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_0\frac{\partial \vec{J}}{\partial t} \quad (1)$$

Manuscript received March 22, 1995. This work was supported in part by NSF through the Research Initiation Award ECS-9309179, and by the Army Research Office under contract #DAAH04-94-G-0243

The authors are with the Department of Electrical Engineering, University of Kentucky, Lexington, KY 40506-0046 USA.

IEEE Log Number 9414130.

where c_0 is the speed of light in free space, σ is the conductivity, and ε_r and μ_r are the relative permittivity and permeabilities. The problem is simplified by defining $\partial\Omega$ to be an electric (PEC) or magnetic (PMC) wall. The inner product of (1) with a testing function $\vec{T}$ is performed, and applying Green's first vector identity leads to the weak form

$$\iiint_\Omega \left[\frac{1}{\mu_r}(\nabla \times \vec{T})\cdot(\nabla \times \vec{E}) + \mu_0\sigma\vec{T}\cdot\frac{\partial \vec{E}}{\partial t} + \frac{\varepsilon_r}{c_0^2}\vec{T}\cdot\frac{\partial^2 \vec{E}}{\partial t^2}\right] \cdot d\Omega = -\iiint_\Omega \mu_0\vec{T}\cdot\frac{\partial \vec{J}}{\partial t}\,d\Omega. \quad (2)$$

The vector fields are expanded using one-form Whitney edge elements $\vec{W}_j$ weighted by constant coefficients e_j that are continuous functions of time. Taking the first variation of (2) and evaluating it at the stationary point leads to the second-order ordinary differential equation

$$[T_\varepsilon]\frac{1}{c_0^2}\frac{d^2 e}{dt^2} + [T_\sigma]\frac{\eta_0}{c_0}\frac{de}{dt} + [S]e = -f \quad (3)$$

where $[T_\varepsilon], [T_\sigma]$ and $[S]$ are time-independent matrices, and

$$[T_\varepsilon]_{i,j} = \iiint_\Omega \begin{Bmatrix}\varepsilon_r\\ \sigma\end{Bmatrix}\vec{W}_i\cdot\vec{W}_j\,d\Omega,$$

$$[S]_{i,j} = \iiint_\Omega \frac{1}{\mu_r}\nabla\times\vec{W}_i\cdot\nabla\times\vec{W}_j\,d\Omega.$$

$$f_i = \iiint_\Omega \mu_0\vec{W}_i\cdot\frac{\partial \vec{J}}{\partial t}\,d\Omega. \quad (4)$$

Based on the Newmark-Beta formulation [8], [9], (3) is approximated as

$$\begin{aligned}&[T_\varepsilon]\frac{1}{(c_0\Delta t)^2}(e^{n+1} - 2e^n + e^{n-1})\\ &\quad + [T_\sigma]\frac{\eta_0}{c_0 2\Delta t}(e^{n+1} - e^{n-1})\\ &\quad + [S](\beta e^{n+1} + (1-2\beta)e^n + \beta e^{n-1}) = 0 \quad (5)\end{aligned}$$

where, e^n is the discrete-time representation of e, namely, $e^n = e(n\Delta t)$ and β is a constant. This leads to the implicit update scheme

$$\begin{aligned}e^{n+1} =& [[T_\varepsilon] + \tfrac{1}{2}\eta_0 c_0\Delta t[T_\sigma] + \beta(c_0\Delta t)^2[S]]^{-1}\\ &\cdot\{[2[T_\varepsilon] - (1-2\beta)(c_0\Delta t)^2[S]]e^n\\ &- [[T_\varepsilon] - \tfrac{1}{2}\eta_0 c_0\Delta t[T_\sigma] + \beta(c_0\Delta t)^2[S]]e^{n-1}\\ &- (c_0\Delta t)^2[\beta f^{n+1} + (1-2\beta)f^{n+1} + \beta f^{n-1}]\}. \quad (6)\end{aligned}$$

1051-8207/95$04.00 © 1995 IEEE

III. Stability Analysis

Equation (6) is expressed in reduced notation as the second-order difference equation

$$e^{n+1} = 2[A]^{-1}[B]e^n - [A]^{-1}[C]e^{n-1} \tag{7}$$

where it is assumed $f^n = 0$. This can be reduced to a first-order equation by introducing the vectors

$$\boldsymbol{y}^{n+1} = \begin{bmatrix} e^{n+1} \\ e^n \end{bmatrix}, \quad \boldsymbol{y}^n = \begin{bmatrix} e^n \\ e^{n-1} \end{bmatrix} \tag{8}$$

and then expressing (7) as

$$\boldsymbol{y}^{n+1} = [M]\boldsymbol{y}^{n-1}; \quad [M] = \begin{bmatrix} 2[A]^{-1}[B] & -[A]^{-1}[C] \\ [I] & 0 \end{bmatrix}. \tag{9}$$

Stability of the first-order equation requires $\rho([M]) < 1$, where $\rho([M])$ is the spectral radius of M.

Define $(\lambda_1, \boldsymbol{x}_1)$ to be an eigensolution of $[A]^{-1}[B]$, and $(\lambda_2, \boldsymbol{x}_2)$ to be an eigensolution to $[A]^{-1}[C]$. It can be shown that $[A]^{-1}[B]$ and $[A]^{-1}[C]$ are simultaneously diagonalizable [10]. This implies that $\boldsymbol{x}_2 = c\boldsymbol{x}_1$, where c is a constant. The eigenspectrum of $[M]$ is then found from the eigenvalue problem

$$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \xi \begin{bmatrix} x \\ y \end{bmatrix}. \tag{10}$$

Let $[P]$ be the eigenvector matrix of $[A]^{-1}[B]$ and $[A]^{-1}[C]$. Then, $[A]^{-1}[B] = [P][D_1][P]^{-1}$ and $[A]^{-1}[C] = [P][D_2][P]^{-1}$, where $[D_1]$ and $[D_2]$ are diagonal matrices containing the eigenvalues of $[A]^{-1}[B]$ and $[A]^{-1}[C]$. Subsequently, (10) can be expressed as

$$\begin{bmatrix} 2[D_1] & -[D_2] \\ [I] & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \xi \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{11}$$

where $x' = [P]^{-1}x$ and $y' = \eta x'$. Equation (11) is then reduced to 2×2 matrices for each pair of eigenvalues λ_1 and λ_2

$$\begin{bmatrix} 2\lambda_1 & -\lambda_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \xi \begin{bmatrix} x' \\ y' \end{bmatrix}. \tag{12}$$

The eigenvalues ξ can then be easily derived from the characteristic equation and have solutions $\xi = \lambda_1 \pm \sqrt{\lambda_1^2 - \lambda_2}$. Stability requires that $|\xi| < 1$, which is true if $|\lambda_2| < 1$, and $|\lambda_1| < (1 + \lambda_2)/2$, or, if $\rho([A]^{-1}[C]) < 1$, and $\rho([A]^{-1}[B]) < (1 + \lambda_2)/2$.

The spectral radius of $[A]^{-1}[C]$ can be determined in closed form if it is assumed that Ω is a homogeneous space. (Note that this is not a requirement for stability.) This results in $[T_\sigma] = \sigma/\varepsilon_r[T_\varepsilon]$. Comparing (6) and (7), $[A]$ and $[C]$ are then expanded in terms of $[T_\varepsilon]$ and $[S]$, and $[C]x - \lambda_2[A]x = 0$ can be rewritten as

$$[S]x = \kappa[T_\varepsilon]x \tag{13}$$

where

$$\kappa = \frac{(\lambda_2 - 1) + (\lambda_2 + 1)\eta_0 c_0 \Delta t \sigma / 2\varepsilon_r}{\beta (c_0 \Delta t)^2 (1 - \lambda_2)}. \tag{14}$$

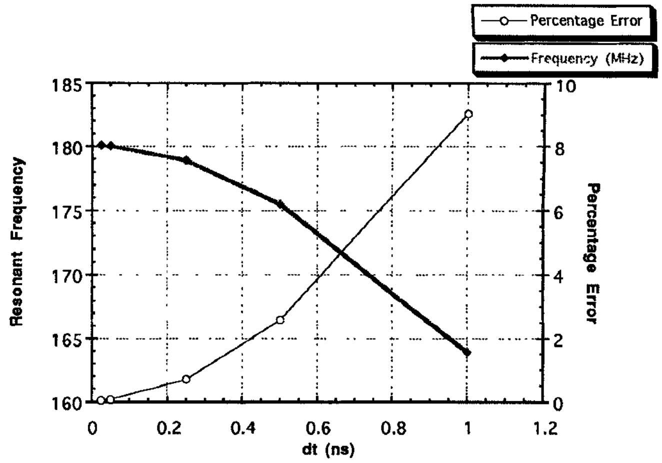

Fig. 1. Numerically derived resonant frequency versus time step ($\beta = 0.25, h = .125$ m). The exact resonant frequency is 180.15 MHz.

It can be shown that $[S]$ is a nonnegative matrix and $[T_\varepsilon]$ is a positive definite matrix, and thus $\kappa \geq 0$. Also, $\kappa_{\max} = \lambda^S_{\max}/\lambda^T_{\min}$, where $\lambda^S_{\max}$ and $\lambda^T_{\min}$ are the maximum and minimum eigenvalues of $[S]$ and $[T_\varepsilon]$, respectively [5]. Then, from (14)

$$\lambda_2 = \frac{1 + \beta\kappa(c_0\Delta t)^2 - \frac{1}{2}\eta_0 c_0 \Delta t \frac{\sigma}{\varepsilon_r}}{1 + \beta\kappa(c_0\Delta t)^2 + \frac{1}{2}\eta_0 c_0 \Delta t \frac{\sigma}{\varepsilon_r}} \tag{15}$$

and $|\lambda_2| < 1$ for all β.

Similarly, the spectral radius of $[A]^{-1}[B]$ is determined from the eigenvalue equation $[B]x - \lambda_1[A]x = 0$, leading to

$$\lambda_1 = \frac{1 - \frac{1 - 2\beta}{2}\kappa(c_0\Delta t)^2}{1 + \beta\kappa(c_0\Delta t)^2 + \frac{1}{2}\eta_0 c_0 \Delta t \frac{\sigma}{\varepsilon_r}} \tag{16}$$

where κ are the eigenvalues of (13). Finally, stability requires $|\lambda_1| < (1 + \lambda_2)/2$, which is true for all $\kappa \geq 0$ if $\beta > 1/4$. Since $k_{\max}$ is finite, then stability can be defined in the weak sense. Thus, choosing

$$\beta \geq \tfrac{1}{4} \tag{17}$$

leads to unconditional stability of the second-order update expression (6).

IV. Numerical Results

A FORTRAN program based on the FETD method presented herein has been implemented on an HP-720 workstation. A matrix solution is performed at each time step using the Conjugate Gradient method with diagonal preconditioning. To validate the accuracy and stability of this method, the simple example of the excitation of a lossless cavity ($\sigma = 0$) with perfectly conducting walls is studied. The dimensions of the cavity are 1.0 m by 0.5 m by 1.5 m, and the cavity is modeled using tetrahedral elements.

Fig. 1 illustrates a plot of the resonant frequency of the cavity for the TE^z_{101} mode versus the discrete time step

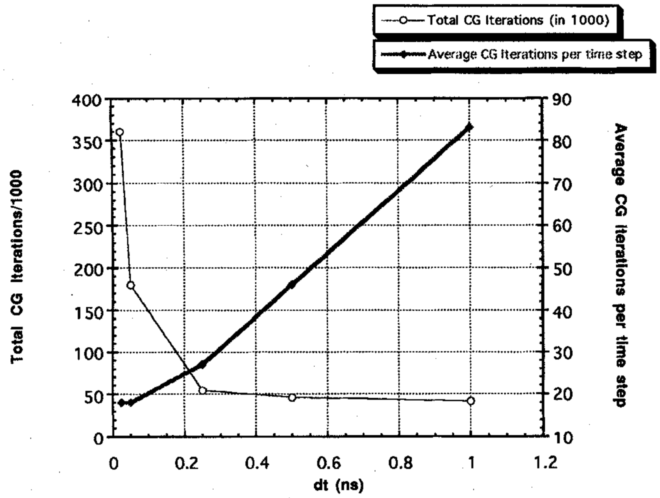

Fig. 2. Computational work required versus time step ($\beta = 0.25, h = .125$ m).

Δt for $\beta = 1/4$. The resonant frequency was obtained by Fourier transforming the time response. The simulations were performed over the time period $t =$ 0–500 ns, and the mesh had an average tetrahedral radius of $h = 0.125$ m. As expected, the error in the computed resonant frequency decreases as the time step is decreased. Note that the algorithm is stable for all time steps. Comparatively, for this mesh a central difference approximation ($\beta = 0$) has a stability criterion of $\Delta t <$ 0.18 ns. Fig. 2 shows the variation of computational effort with the time step. This plot shows that number of iterations of the conjugate gradient method required per time step decreases as the time step becomes smaller. However, the total computational work required actually increases as the time step is decreased since a larger number of time steps is required to reach steady state as illustrated in Fig. 2. An optimal time step of roughly 0.25 ns will provide both accuracy and lower computational time.

Fig. 3 illustrates the variation of the resonant frequency for the TE^z_{101} and TE^z_{102} modes with β. (The exact resonant frequency for the TE^z_{102} mode is 249.83 MHz.) for a time step of 0.1 ns (this was chosen such that $\beta = 0$ was stable). These figures indicate that the error increases with β. Thus, the optimal choice for β is 1/4.

V. Conclusion

A finite element time-domain algorithm was presented. Using the Newmark-Beta formulation, it was shown that an unconditionally stable scheme is achievable, providing the interpolation parameter $\beta \geq 1/4$. It was further shown that choosing $\beta = 1/4$ minimized solution error. Finally, it has

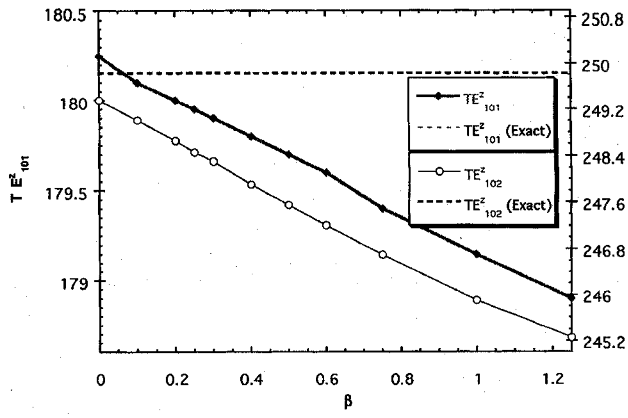

Fig. 3. Resonant freuqency versus β ($\Delta t = 0.1$ ns, $h = 0.125$ m).

been found, that accurate solutions are obtained for time steps $\Delta t \approx (\lambda/15)/c$, where c is the speed of light, and λ is the maximum wavelength of interest in the frequency response.

Acknowledgment

The authors would like to thank the Center for Computational Sciences at the University of Kentucky for their partial support of Mr. Navsariwala through a CCS fellowship and Dr. Z. Bai for his useful conversations on eigenvalue analysis.

References

[1] A. Taflove, *Computational Electrodynamics, the Finite-Difference Time-Domain Method.* Boston, MA: Artech House, 1995.
[2] R. L. Lee and N. K. Madsen, "A mixed finite element formulation for Maxwell's equations in the time domain," *J. Computational Phys.*, vol. 88, no. 2, pp. 284–304, 1990.
[3] P. B. Monk, "A mixed method for approximating Maxwell's equations," *SIAM J. Numerical Analysis,* vol. 28, pp. 1610–1634, Dec. 1991.
[4] K. Mahadevan and R. Mittra, "Radar cross section computation of inhomogeneous scatterers using edge-based finite element methods in frequency and time domains," *Radio Sci.,* vol. 28, pp. 1181–1193, Nov.–Dec. 1993.
[5] J.-F. Lee, "WETD—A finite element time-domain approach for solving Maxwell's equations," *IEEE Microwave and Guided Wave Lett.,* vol. 4, pp. 11–13, Jan. 1994.
[6] J. T. Elson, H. Sangani, and C. H. Chan, "An explicit time-domain method using three-dimensional Whitney elements," *Microwave and Opt. Technol. Lett.,* vol. 7, no. 13, pp. 607–609, 1994.
[7] S. D. Gedney and U. Navsariwala, "A Comparison of the performance of the FDTD, FETD, and planar generalized Yee algorithms on high performance parallel computers," *Int. J. Numerical Modeling (Electronic Networks, Devices and Fields),* vol. 8, pp. 265–276, May–Aug. 1995.
[8] N. M. Newmark, "A method of computation for structural dynamics," *J. Engineering Mechanics Division, ASCE,* vol. 85, pp. 67–94, July 1959.
[9] O. C. Zienkiewicz, "A new look at the Newmark, Houboult and other time stepping formulas. A weighted residual approach," *Earthquake Engineering and Structural Dynamics,* vol. 5, pp. 413–418, 1977.
[10] S. Gedney and U. Navsariwala, "An unconditionally stable finite-element time-domain analysis of lossy printed waveguiding structures," Univ. of Kentucky Dept. of Electrical Engineering Tech. Rep. #EMG-95-001, Univ. of Kentucky, Lexington, KY, Jan. 1995.